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OPTIMAL DESIGN OF STRUCTURES WITH BUCKLING CONSTRAINTS

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Abstract—The paper presents an iterative, finite element method for minimum weight design of structures with respect to buckling constraints. The redesign equation is derived from the optimality criterion, as opposed to a numerical search procedure, and can handle problems that are characterized by the existence of two fundamental buckling modes at the optimal design. Application of the method is illustrated by beam and orthogonal frame design problems.

1. INTRODUCTION

THE present paper treats the following optimal design problem: given the structural layout, the load distribution, and certain constraints on the design variables, find the minimum weight structure that has a prescribed buckling strength. The design variables, which specify the sizes of the structural members, are assumed to be related linearly to the structural weight (e.g. cross-sectional area for a beam, thickness for a plate).

The optimal design problem on the whole is nonlinear, due to the nonlinear character of the constraints (buckling load-element size relations), which precludes analytical solutions in all but a few simple cases. Consequently we find that the published solutions deal exclusively with the design of columns. The first design in this category—the optimal shape of a simple supported column—is attributed to Clausen [1]; the result was later discovered independently by Keller [2]. Tadjbakhsh and Keller [3] extended the work to columns with different supports and gave the first proof of the optimal character of the design. A reformulation of the optimality criteria from energy concepts, rather than the incremental equilibrium equations, was presented by Taylor [4], and Prager and Taylor [5]. The last reference, together with Taylor and Liu [6], also introduced minimum cross-sectional area constraints into the formulation. The effect of self-weight on the optimal design has been investigated by Keller and Niordson [7], and Huang and Sheu [8].

The mathematical difficulties that arise in more complex structures leave little doubt that numerical methods offer the only practical, and in most cases the only workable means of optimal design. Computer algorithms using the finite element concept are particularly attractive, due to the great popularity of finite element analysis, and the availability of an extensive library of related subroutines.

In the present paper we propose such a finite element method for design of elastic structures. A few aspects of the technique are similar to the programs developed by Rubin [9], and Venkayya *et al.* [10] for a related problem—optimal design with respect to a prescribed fundamental frequency. There are, however, significant differences, which make the present approach applicable to a much greater variety of problems than would be possible with a straightforward adaptation of [9] or [10] to buckling-constrained design.

The development of the design equations is based on two simplifying assumptions. Firstly, the internal forces acting in the structure prior to buckling are taken to be statically determinate. Secondly, it is presumed that buckling deformations caused by direct stresses can be neglected in cases where the element size-bending stiffness relations are nonlinear. The applicability of the equations is thus confined to structures with linear size–stiffness relations, or to beams, plates and orthogonal frames.

The above assumptions were chosen mainly for the sake of computational economy, and not for reasons of insurmountable mathematical complexity. It would not be difficult to relax the restrictions, but at the cost of much greater computation times and storage requirements. In view of the main objective of this report—to demonstrate the feasibility of the basic optimization technique—the extra cost was not considered justified.

2. BASIC EQUATIONS

Size–stiffness relations

Due to manufacturing considerations, it is generally desirable to keep the design variables A_i (i denotes the element number) constant within each element. The weight of the element can then be written as

$$W_i = \rho_i A_i, \tag{1}$$

where ρ_i is the unit weight (weight per unit length, or per unit area of middle surface). If the design objective is to minimize the total cost, rather than weight, then ρ_i should be interpreted as the unit cost.

We assume that the elastic stiffness matrix $[K_i]$ of a typical (ith) element has the form

$$[K_i] = [k_i]A_i^m, \tag{2}$$

where the unit stiffness matrix $[k_i]$ is independent of A_i. The above size-stiffness relationship, although rather restrictive, does contain several important cases: $m = 1$ corresponds to thin-walled beams or sandwich plates, where the wall thickness is the design variable, or to structures composed of elements that carry direct stresses only; beam elements with constant cross-sectional shape, but variables area, can be treated with $m = 2$; $m = 3$ is applicable to plates of variable thickness.

With the exception of $m = 1$, (2) is capable of representing bending or torsional stiffness only, thus excluding buckling of shells and nonorthogonal frames, where the extensional deformations are not negligible. A more general size–stiffness relationship would be

$$[K_i] = \sum_{m=0}^{3} [k_i^{(m)}]A_i^m,$$

but it turns out that each $[k_i^{(m)}]$ must be stored separately for each element. This facility is lacking in presently available analysis programs.

Although we presume that each A_i is continuously variable, it may be possible to apply the design technique to standard structural sections, since their stiffness properties can sometimes be approximated by (2) (cf. Ref. [11]).

From a practical viewpoint, it is essential that the design algorithm permit us to assign the structural elements into groups, such that each element in a group has the same value of A_i. The need for equal size groups arises when a member has to be divided into several finite elements for the purpose of analysis, or when it is desirable to limit the total number of different member sizes.

Equal size constraints can be handled by introducing group stiffness matrices $[K_g]$, obtained by assembling the stiffness matrices of all the elements that belong to group g. The rules for performing the assembly are based, as usual, on the invariance of strain energy:

$$\{u_g\}^T[K_g]\{u_g\} = \sum_{i \in g} \{u_i\}^T[K_i]\{u_i\},$$

where $\{u_g\}$ and $\{u_i\}$ represent the generalized displacement vectors for the group g, and the ith elements respectively. Henceforth, there is no need to distinguish between problems with equal area constraints, and those without the constraints. All the formulae that follow are valid for both types of problems, provided that A_i, $\{u_i\}$, $[K_i]$ etc. are interpreted as belonging to the ith group if equal size constraints are being used.

Optimality criterion

It is assumed that the loads acting on the structure can be considered to be proportional to a single load parameter P. The critical values of P (values that correspond to buckling) are denoted by P_r, and are presumed to be arranged in an ascending order: $P_1 \leq P_2 \leq \ldots$. If we use the notation P^* for the smallest allowable critical value of the load parameter, the behavioral constraint can be expressed as

$$P_r \geq P^*, \qquad r = 1, 2 \ldots R, \tag{3}$$

where R equals the number of degrees of freedom of the finite element model.

In addition to the limits on the buckling strength, we also introduce minimum size constraints on the design variables:

$$A_i \geq A_i^*, \qquad i = 1, 2 \ldots I, \tag{4}$$

where A_i^* are the prescribed minimum values, and I denotes the number of independent design variables.

The inequalities (3) and (4) can be replaced by the equality constraints

$$P_r - p_r^2 = P^*, \qquad A_i - a_i^2 = A_i^*, \tag{5}$$

which are more convenient to use in the derivation of the optimality criterion. In (5), p_r and a_i must be interpreted as unknown variables, free from constraints.

The design objective is to minimize the total structural weight

$$W = \sum_{i=1}^{I} W_i = \sum_{i=1}^{I} \rho_i A_i \tag{6}$$

subject to the equality constraints (5). The equivalent problem in calculus of variations is to find the values of A_i, a_i and p_r that make the function

$$V = \sum_{i=1}^{I} \rho_i A_i - \sum_{r=1}^{R} \lambda_r (P_r - p_r^2) - \sum_{i=1}^{I} \mu_i (A_i - a_i^2) \tag{7}$$

stationary, where λ_r and μ_i are non-negative constants (Lagrangian multipliers). The operations $\partial V/\partial A_i = 0$, $\partial V/\partial p_r = 0$ and $\partial V/\partial a_i = 0$ yield, respectively

$$\rho_i - \sum_{r=1}^{R} \lambda_r P_{r,i} - \mu_i = 0, \tag{8a}$$

$$\lambda_r p_r = 0, \tag{8b}$$

$$\mu_i a_i = 0. \tag{8c}$$

The notation $(\ )_{,i} = \partial/\partial A_i$ was used in (8a).

In view of (5), (8b, c) may be expressed in the form

$$\lambda_r \begin{cases} \geq 0 & \text{if } P_r = P^* \\ = 0 & \text{if } P_r > P^* \end{cases} \tag{9a}$$

and

$$\mu_i \begin{cases} \geq 0 & \text{if } A_i = A_i^* \\ = 0 & \text{if } A_i > A_i^*. \end{cases} \tag{9b}$$

Consequently, the optimality criterion (8a) becomes

$$\rho_i - \sum_{r=1}^{\tilde{R}} \lambda_r P_{r,i} \begin{cases} = 0 & \text{if } A_i > A_i^* \\ \geq 0 & \text{if } A_i = A_i^*, \end{cases} \tag{10}$$

where $\tilde{R}$ is the number of non-zero Lagrangian multipliers λ_r, i.e. the number of active buckling constraints $P_r = P^*$.

Experience with a variety of problems seems to indicate that it is sufficient to take $\tilde{R} \leq 2$; that is, the optimal design is determined either by the first buckling mode alone, or by the first and second modes simultaneously. Theoretically, it is possible for more than two active buckling constraints to exist, but no such examples were encountered in the use of the optimization program. In this connection it should be noted that Refs. [9] and [10] allow for only one active frequency constraint.

Derivatives of the load parameter

Buckling is governed by the characteristic value problem

$$[K]\{u\} = P[H]\{u\} \tag{11}$$

where $[K]$ and $[H]$ are the elastic and geometric stiffness matrices of the structure, respectively, and $\{u\}$ is the generalized displacement vector. The gradients of the critical values of P are easily obtainable from (11) (cf. Ref. [12], pp. 244–245):

$$P_{r,i} = \frac{\{u^{(r)}\}^T([K_{,i}] - P_r[H_{,i}])\{u^{(r)}\}}{\{u^{(r)}\}^T[H]\{u^{(r)}\}}, \tag{12}$$

where $\{u^{(r)}\}$ represents the buckling mode associated with P_r.

The components of the geometric stiffness matrix are linear functions of the internal forces that act in the prebuckling state. In the present paper we assume that these forces are statically determinate, in which case $[H_{,i}]$ vanishes. We would like to add that it is possible to handle statically indeterminate problems since the derivatives of the internal forces with respect to A_i are calculable (cf. Ref. [12], pp. 242–243). The computations, however, require the solution of a set of simultaneous equations, resulting in considerably longer computer run times.

In calculating the derivatives of the elastic stiffness matrix, we note that only the ith element contributes to $[K_{,i}]$. Thus, in view of (2) we get

$$[K_{,i}] = [K_{i,i}] = m[K_i]/A_i, \tag{13}$$

and (12) becomes

$$P_{r,i} = mU_i^{(r)}/A_i, \tag{14}$$

where we used the abbreviation

$$U_i^{(r)} = \frac{\{u_i^{(r)}\}^T[K_i]\{u_i^{(r)}\}}{\{u^{(r)}\}^T[H]\{u^{(r)}\}}. \tag{15}$$

If the buckling modes are normalized with respect to $[H]$, $U_i^{(r)}$ equals the strain energy of the rth buckling mode stored in the ith element.

Substituting the results in (10), the optimality criterion becomes

$$(m/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)} \begin{cases} = A_i & \text{if } A_i > A_i^* \\ \leq A_i & \text{if } A_i = A_i^*. \end{cases} \tag{16}$$

3. EQUATIONS USED IN DESIGN ALGORITHM

Redesign equation

The basic function of the optimization algorithm used in the present paper is to solve (16) by the method of successive, linearized iterations, starting from some initial design. We could adopt (16) directly as the redesign equation:

$$A_i' = \begin{cases} C_i & \text{if } C_i > A_i^* \\ A_i^* & \text{if } C_i \leq A_i^* \end{cases} \tag{17}$$

where

$$C_i = (m/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)},$$

and A_i' denotes the new, improved design, while the unprimed quantities refer to the current design.

It turns out that the convergence of the iterative process can be considerably improved by introducing the relaxation factor ω, and replacing the expression for C_i by

$$C_i = \omega A_i + (1-\omega)(m/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)}. \tag{18}$$

The best starting value of the relaxation factor has been found to be

$$\omega = m/(m+1). \tag{19}$$

Values different from (19) are needed only in problems with special convergence difficulties, and then only if the structural weight is to be computed within 1 or 2 per cent of the true optimal weight.

An alternative derivation of the redesign equation, which provides detailed physical justification of (18) and (19), is given in the Appendix.

The Lagrangian multipliers λ_r, $r = 1, 2 \ldots \tilde{R}$, are determined from the condition $P_r' = P^*$, $r = 1, 2 \ldots \tilde{R}$, i.e. from the requirement that the new design A_i' must be critical. The change in P_r due to the design change δA_i can be estimated from the linear approximation

$$\delta P_r = \sum_{i=1}^{I} P_{r,i}\delta A_i = m \sum_{i=1}^{I} U_i^{(r)}\delta A_i/A_i, \tag{20}$$

where we substituted (14) for $P_{r,i}$.

It is convenient to divide δP_r into two parts: $\delta P_r = (\delta P_r)_{\text{pass}} + (\delta P_r)_{\text{act}}$. The first part

$$(\delta P_r)_{\text{pass}} = m \sum_{i_{\text{pass}}}^{I} U_i^{(r)}(A_i^* - A_i)/A_i \tag{21}$$

contains the contribution of the passive elements; that is, elements governed by minimum size constraints after the redesign. The second part

$$(\delta P_r)_{\text{act}} = m \sum_{i_{\text{act}}}^{I} U_i^{(r)}(C_i - A_i)/A_i$$

represents the change due to the active elements.

We now replace C_i by (18), set $\delta P_r = P^* - P_r$, and obtain the following set of simultaneous equations for the Lagrangian multipliers

$$m^2(1-\omega) \sum_{s=1}^{\tilde{R}} \left(\lambda_s \sum_{i_{\text{act}}}^{I} U_i^{(s)} U_i^{(r)}/W_i \right) = m(1-\omega) \sum_{i_{\text{act}}}^{I} U_i^{(r)} - m \sum_{i_{\text{pass}}}^{I} U_i^{(r)}(A_i^* - A_i)/A_i + P_r^* - P_r, \tag{22}$$

$r = 1, 2 \ldots \tilde{R}$. As noted before, it appears sufficient to consider only the possibility of $\tilde{R} = 1$ or $\tilde{R} = 2$.

Equations (22) can be solved only if $\tilde{R}$ and the identities of active and passive members are known beforehand. As this is generally not the case, a trial-and-error procedure, outlined in Fig. 1, was used in the program. The method, which is similar to that developed in Ref. [13] for displacement-constrained design, has proven to be efficient, requiring only a few iterations.

Scaling operation

In certain problems, such as structures with elastic supports, repeated applications of the redesign equations (17), (18) and (22) may result in a sequence of designs that do not converge. The difficulty can be traced to the approximate nature of the expression for changes in buckling loads (20). It has been observed that the predicted values of δP_r may be grossly in error when δA_i are obtained from the redesign equations. On the other hand, (20) appears to give satisfactory results if the changes in element sizes are more-or-less uniform.

In view of the last observation, we introduce the uniform scaling operation

$$A_i' = CA_i \tag{23}$$

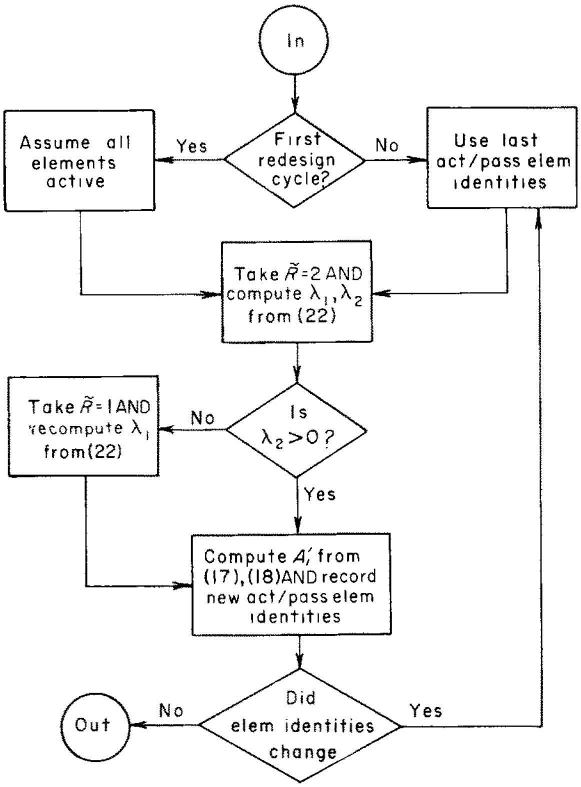

FIG. 1. Flow diagram of redesign operation.

(C is the scale factor), which is used whenever the current value of P_1 departs from P^* by more than a prescribed amount ε. Redesign equations are applied only if P_1 is within the acceptable band $P^* - \varepsilon \leq P_1 \leq P^* + \varepsilon$ (see Fig. 2).

The scaling operation is desirable even in problems that would converge to the optimal design by the use of the redesign equations alone, since it helps the designer to monitor the progress of the design process. The weight of successive, scaled designs gives a good indication of the convergence characteristics of the problem, enabling us to stop the design process and take corrective measures whenever the weight changes begin to behave in an undesirable manner.

The scale factor C in (23) is determined from the requirement $(P_r + \delta P_r)_{\min} = P^*$. The changes in the buckling loads could be computed from (20), but the special form of the size–stiffness relations (2) enables us to use a more accurate estimate:

$$\delta P_r = a_r + b_r C^m. \tag{24}$$

Since we must have $\delta P_r = 0$ when $C = 1$ (no design change), it follows that $b_r = -a_r$. The condition that (24) and (20) should yield the same results for infinitesimal design changes gives us the second equation for a_r and b_r:

$$a_r(1 - C^m) = m \sum_{i=1}^{I} U_i^{(r)}(C-1) \quad \text{as} \quad C \to 1,$$

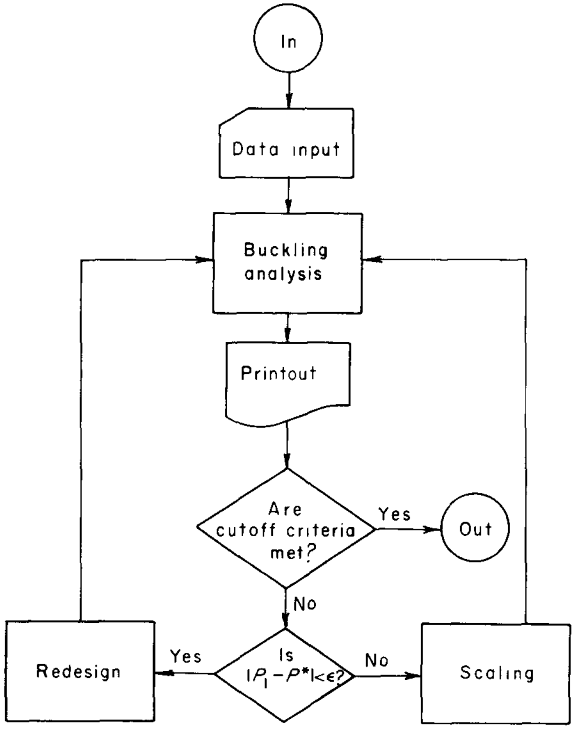

FIG. 2. Flow diagram of optimization algorithm.

where $\delta A_i/A_i = C-1$ was substituted in (20). Hence

$$a_r = m \lim_{C\to 1}\left(\frac{C-1}{1-C^m}\right)\sum_{i=1}^{I} U_i^{(r)} = -\sum_{i=1}^{I} U_i^{(r)}$$

and (24) becomes

$$\delta P_r = (C^m-1)\sum_{i=1}^{I} U_i^{(r)}. \tag{25}$$

Setting $(P_r+\delta P_r)_{\min} = P^*$, (25) yields

$$C^m = \max_r \left[\left(P^* - P_r + \sum_{i=1}^{I} U_i^{(r)}\right)\Big/\sum_{i=1}^{I} U_i^{(r)}\right], \quad r = 1,2\ldots \tilde{R}. \tag{26}$$

If the strain energy of the system is stored entirely within the structural elements (e.g. in the absence of elastic supports), then, according to (15), $\sum_i U_i^{(r)}$ represents the Rayleigh quotient, and is, therefore, equal to P_r. In this case $C^m = P^*/(P_r)_{\min} = P^*/P_1$, resulting in an exact scaling operation, i.e. equation (25) will predict changes in P_r exactly, while the buckling modes remain unchanged. Consequently, a reanalysis of the scaled structure is unnecessary.

The preceding discussion is generally applicable only when the structure is to be scaled up ($C > 1$). If the element sizes are to be reduced, the scaling operation must be modified such as not to violate the minimum size constraints.

For downward scaling, we change (23) to

$$A_i' = \begin{cases} CA_i & \text{if } CA_i \geq A_i^* \\ A_i^* & \text{if } CA_i < A_i^*, \end{cases} \tag{27}$$

and write, as in the previous section, $\delta P_r = (\delta P_r)_{\text{pass}} + (\delta P_r)_{\text{act}}$. The contribution of active members is again calculated from (25) (note that now the sum is to be taken over the active members only), whereas $(\delta P_r)_{\text{pass}}$ is obtained from (21). As a result, the equation for the scale factor becomes

$$C^m = \max_r \left[\frac{P_r^* - P_r - m \sum_{i_{\text{pass}}} U_i^{(r)}(A_i^* - A_i)/A_i + \sum_{i_{\text{act}}} U_i^{(r)}}{\sum_{i_{\text{act}}} U_i^{(r)}} \right], \qquad r = 1, 2 \ldots \tilde{R}. \tag{28}$$

The identities of the active and passive elements are established by the same trial-and-error procedure that was used in redesign.

Downward scaling generally occurs only as the first operation in the design processes in cases where the stiffness of the initial design is too large. The redesign operation usually causes P_1 to fall below P^*, so that all subsequent scaling operations involve increases in element sizes.

4. EXAMPLES

General remarks

A simplified flow diagram of the computer program used in conjunction with the preceding theory appears in Fig. 2. The buckling analysis was carried out with an iterative Rayleigh–Ritz method, described in detail in Ref. [14]. The computations were confined to two characteristic values and mode shapes.

The design process was terminated whenever one of the following two cutoff criteria was met:

(i) the number of redesign cycles exceeded a prescribed number; or

(ii)

$$P^* - \varepsilon \leq P_1 \leq P^* + \varepsilon, \tag{29a}$$

and

$$1 - \delta \leq (m/W_i) \sum_{r=1}^{\tilde{R}} \lambda_r U_i^{(r)} < 1 + \delta \qquad (i_{\text{act}}). \tag{29b}$$

The last inequality corresponds to the optimality criterion (16).

In all the examples we used the rather stringent values $\varepsilon = 0{\cdot}01P^*$ and $\delta = 0{\cdot}05$. The value of the relaxation factor ω was obtained from (19), unless specified otherwise.

It was found desirable to monitor the design algorithm by limiting the prescribed number of redesigns to three or four. This is usually sufficient for problems with good convergence characteristics to reach the cutoff criteria. If the convergence is poor, three to four cycles are adequate for making a diagnosis, and taking corrective measures (e.g. adjusting the value of ω) before restarting the program.

Columns

The first example involves a cantilever column of length l, subjected to a dead load P at the free end. The data used in the problem was $P^* = 5000$ lb, $l \doteq 100$ in., $E = 10^7$ psi and $I_i = 1{\cdot}0A_i$ (I_i is the moment of inertia of the ith element). The column was divided into 10 finite elements, and the cross-sectional area of each element was considered as an independent design variable. No limits were placed on the areas. A constant cross-sectional area of 1·0 in^2 was chosen as the initial design.

Four successive, acceptable designs (designs for which P_1 lies within the acceptable band) produced by the computer program are shown in Fig. 3. Although each element of the column is prismatic, the figures were obtained by plotting the cross-sectional area at the mid-point of each element, and joining the points with a smooth curve.

The cutoff criteria (29a, b) were satisfied after four redesigns; however, the shape of the column following the third redesign (Fig. 3d) was indistinguishable from the exact analytical solution [5].

The optimal design of the cantilever column in Fig. 3 was governed by a single (fundamental) buckling mode. Figure 4 presents an example where the optimization

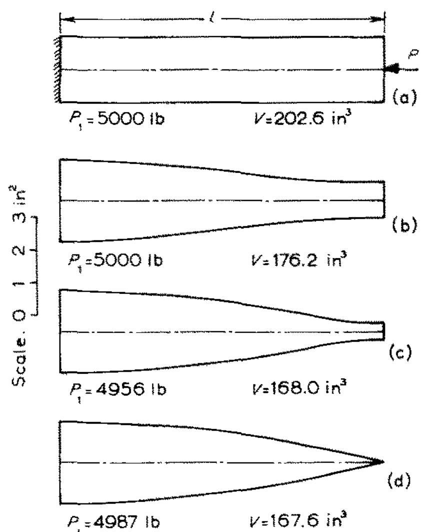

FIG. 3. Successive acceptable designs in the optimization of a cantilever column.

must be carried out with respect to two modes. The beam in question has simple end supports, and rests on an elastic foundation with a spring constant of $\beta = 2{\cdot}5$ lb/in/in. The remaining data was $P^* = 12{,}500$ lb, $l = 200$ in., $E = 10^7$ psi and $I_i = 1{\cdot}0A_i$. Twenty finite elements of equal length were used, and again there was no constraints on the cross-sectional areas. The initial design was deliberately made asymmetric about the mid-point of the column.

The third redesign (Fig. 4d), although still somewhat asymmetric, has a volume that is already within 0·5 per cent of the true optimal volume (obtained by continuing the design process). The analytical solution to this problem appears to be intractable.

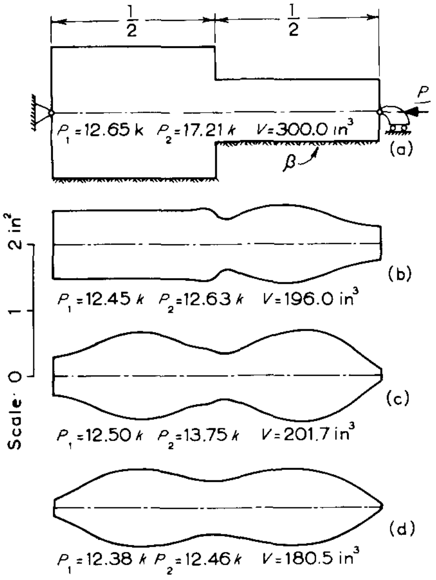

FIG. 4. Successive acceptable designs in the optimization of a simply supported beam on an elastic foundation.

Three-member frame

Apart from a few column problems that can be solved analytically, optimal designs can be readily established by graphical means for structures that have only a few design variables, in which case functional relationships between the buckling loads and the design variables can be established, and plotted in two-dimensional space. One such structure is shown in Fig. 5a—a plane, orthogonal frame, consisting of three prismatic members. Due to the symmetry of loading and structural layout, the cross-sectional areas of the columns will be equal at the optimal design, leaving us with two design variables.

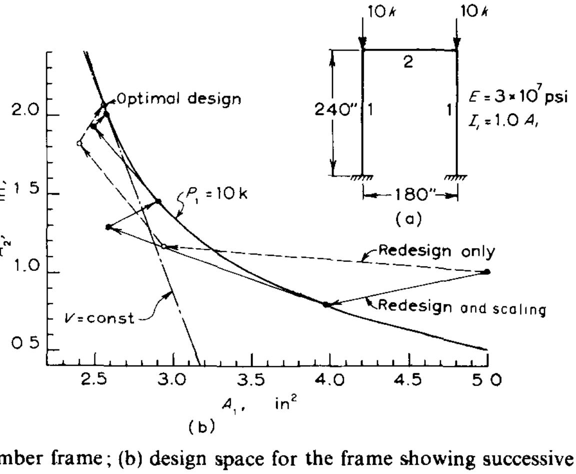

FIG. 5. (a) Three-member frame; (b) design space for the frame showing successive computer designs.

Figure 5b represents the so-called design space for the frame, where each point of the space represents a specific design. The critical designs (designs for which $P_1 = P^*$) trace a line that separates the design space into two regions. The designs above the line are safe against buckling, the points below are unsafe. The optimal design, which in this case is determined by P_1 alone, is represented by the point where a weight (or volume) contour is tangential to the critical design line.

In checking the results of the computer algorithm, we chose an initial design that is well-removed from the optimal point. Two sets of computations were made: one used the scaling operation in conjunction with redesign, the other employed the redesign equations only. Both sets converged to the optimal design with three redesign operations, as indicated in Fig. 5b.

We should add that in order to keep the buckling analysis accurate, each of the column members was divided into four finite elements, and subjected to equal size constraints.

In Fig. 6a we have added a pair of elastic supports that increase the buckling load associated with the asymmetric mode, P_1, which governed the last design. The buckling load of the symmetric mode, P_2, is not affected by the supports. As a result, the optimal design now lies at the intersection of the $P_1 = P^*$ and $P_2 = P^*$ constraint lines (see Fig. 6b).

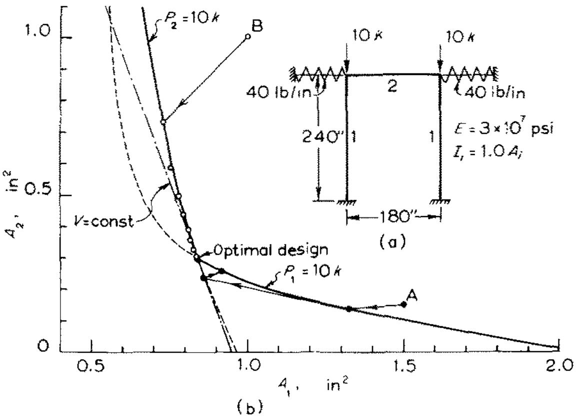

FIG. 6. (a) Three-member frame with spring supports; (b) design space for the frame showing successive computer designs.

When the computer algorithm used the initial design represented by point A in Fig. 6b, the cutoff criteria were satisfied after three redesigns. On the other hand, it took eight redesign operations to reach the optimal design from the starting point B. We have found that this slow, monotonic convergence is typical in regions of the design space where the trace of critical designs is almost parallel to the constant weight contours. Of course, this situation also means that each critical design has a total weight close to the optimal one, although the distribution of weight may be considerably different from the final design. It is usually possible to diagnose the problem after a few redesign cycles, after which the rate of convergence can be increased by restarting the design with a smaller value of the relaxation parameter ω.

Four-story frame

Another type of convergence difficulty is encountered in structures where the lowest passive buckling load ($P_r > P^*$) is close to an active critical load ($P_r = P^*$). We find that the buckling modes become very sensitive to small changes of the design variables near the optimal design, causing the design variables to "overshoot" their optimal values with each redesign. This difficulty can be alleviated by under-relaxation, i.e. by the use of a larger value of ω.

An example on the use of under-relaxation is presented by the design of the frame shown in Fig. 7a. The design data is: $P^* = 750$ lb, $E = 30 \times 10^6$ psi and $I_i = 15{\cdot}9A_i^2$. Minimum cross-sectional area constraints $A_i^* = 0{\cdot}04$ in^2 were used on the vertical members, and 0·02 in^2 on the horizontal beams. In addition, equal size constraints were employed to enforce symmetry on the structure.

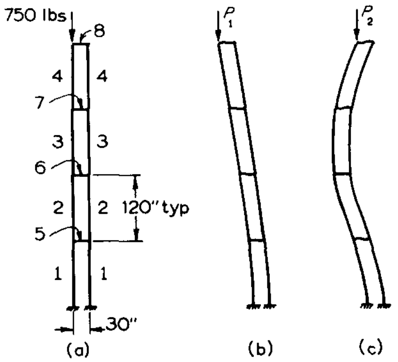

FIG. 7. (a) Four-story frame; (b) first buckling mode; (c) second buckling mode

The optimal design was governed entirely by the first buckling mode (Fig. 7b), but the buckling load of the second mode (Fig. 7c) was only 6 per cent higher than the first.

Successive critical designs obtained from the computer program are given in Table 1. The first two redesigns, using $\omega = 0{\cdot}65$, showed "normal" changes in the design variables.

TABLE 1. SUCCESSIVE ACCEPTABLE DESIGNS IN THE OPTIMIZATION OF THE FOUR-STORY FRAME

Critical design no.	ω	Design variables (sq. in.) A_1–A_4^*	A_5	A_6	A_7	A_8	Buckling loads (lb) P_1	P_2	Total Volume (cu. in.)
1	0·65	0·06272	0·02281	0·02281	0·02281	0·02281	750·0	1081·0	62·95
2	0·65	0·04546	0·02529	0·03021	0·03127	0·02304	750·0	852·7	47·19
3	0·65	0·04258	0·03432	0·03811	0·03391	0·02133	750·0	865·7	44·71
4	0·65	0·04163	0·03176	0·03299	0·03493	0·02831	750·0	838·1	43·81
5	0·90	0·04027	0·03302	0·03772	0·03708	0·02719	750·0	808·0	42·71
6	0·90	0·04000	0·03241	0·03857	0·03862	0·02683	748·7	798·7	42·49
7	0·90	0·04000	0·03248	0·03905	0·03889	0·02617	749·6	797·0	42·50
8	0·90	0·04000	0·03217	0·03911	0·03935	0·02599	749·8	795·4	42·50
9	0·90	0·04000	0·03225	0·03921	0·03946	0·02575	749·9	794·9	42·50

* Governed by minimum size constraints.

The third design produced a sign change in δA_i for three of the four active members, which is symptomatic of design variables overshooting their optimal values. After the program was restarted with $\omega = 0{\cdot}90$, it took five additional redesigns to satisfy the cutoff criteria. The last four weight changes, however, were negligible.

5. CLOSURE

The main features of the optimization technique presented in this paper are:

(a) Rapid rate of weight reduction during the first few redesign cycles. Three redesigns are generally sufficient to reach a weight that differs from the optimal weight by 3 per cent, or less. Moreover, the rate of convergence is independent of the number of design variables, making the methods usable for large structures.

(b) An adjustable relaxation factor that enables us to get as close to the optimal design as we please in problems with abnormal convergence characteristics.

(c) Capability for designing structures where the optimal design is governed by two or more buckling modes simultaneously.

(d) Ability to handle nonlinear size–stiffness relations of structural elements.

(e) Allowance for minimum size, as well as equal size constraints on the design variables.

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REFERENCES

[1] T. CLAUSEN, Über die Form architektonischer Säulen. *Bulletin physico-mathématique de l'Académie* **9**, 368 (1851).
[2] J. B. KELLER, The shape of the strongest column. *Archs. ration. Mech. Analysis* **5**, 275 (1960).
[3] I. TADJBAKHSH and J. B. KELLER, Strongest columns and isoperimetric inequalities for eigenvalues. *J. appl. Mech.* **29**, 159 (1962).
[4] J. E. TAYLOR, The strongest column: an energy approach. *J. appl. Mech.* **34**, 486 (1967).
[5] W. PRAGER and J. E. TAYLOR, Problems of optimal structural design. *J. appl. Mech.* **35**, 102 (1968).
[6] J. E. TAYLOR and C. Y. LIU, Optimal design of columns. *AIAA Jnl* **6**, 1497 (1968).
[7] J. B. KELLER and F. I. NIORDSON, The tallest column. *J. Math. Mech.* **16**, 433 (1966).
[8] N. C. HUANG and C. Y. SHEU, Optimal design of an elastic column of thin-walled cross section. *J. appl. Mech.* **35**, 285 (1968).
[9] C. P. RUBIN, Minimum weight design of complex structures subject to a frequency constraint. *AIAA Jnl* **8**, 923 (1970).
[10] V. B. VENKAYYA, N. S. KHOT, V. A. TISCHLER and R. F. TAYLOR, Design of optimum structures for dynamic loads. Third Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio (1971).
[11] D. M. BROWN and A. H.-S. ANG, Structural optimization by nonlinear programming. *J. Struct. Div., ASCE* **90**, 319 (1966).
[12] R. L. FOX, Optimization Methods For Engineering Design. Addison-Wesley (1971).
[13] R. A. GELLATLY, L. BERKE and W. GIBSON, The use of optimality criteria in Automated Structural Design. Third Conference on Matrix Methods in Structural Mechanics, Wright-Patterson Air Force Base, Ohio (1971).
[14] J. R. ADMIRE, Modal analysis of structures by an iterative Rayleigh–Ritz technique. Research Achievements Review, Vol. 4, Report No. 1, NASA TM X-64528 (1970).

APPENDIX

Let $U_i^{(r)}$ be associated with the current design A_i, and $U_i^{(r)}+\delta U_i^{(r)}$ with the improved design $A_i+\delta A_i$. If the new design is to be the optimal one, it must satisfy the optimality criterion (16):

$$(m/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r(U_i^{(r)}+\delta U_i^{(r)}) = A_i+\delta A_i. \tag{A1}$$

Since we dropped the inequality sign from (16), (A1) is valid for active members only.

In order to obtain an expression for $\delta U_i^{(r)}$ in terms of δA_i, we rewrite (15):

$$U_i^{(r)} = \{f_i^{(r)}\}^T\{u_i^{(r)}\}/G^{(r)}, \tag{A2}$$

where

$$\{f_i^{(r)}\} = [K_i]\{u_i^{(r)}\} \tag{A3}$$

is the vector of nodal forces due to buckling in the rth mode, and $G^{(r)} = \{u^{(r)}\}^T[H]\{u^{(r)}\}$. Restricting $\delta A_i/A_i$ to small values, so that $\delta U_i^{(r)}$ can be interpreted as the first variation of $U_i^{(r)}$, (A2) yields

$$\delta U_i^{(r)} = (\{\delta f_i^{(r)}\}^T\{u_i^{(r)}\}+\{f_i^{(r)}\}^T\{\delta u_i^{(r)}\})/G^{(r)}. \tag{A4}$$

We next assume that the changes in the internal forces produced by redesign have a smaller order of magnitude than the corresponding changes in the nodal displacements. This is a commonly used postulate in optimization with respect to stress or displacement constraints (cf. Ref. [13]). Setting $\{\delta f_i^{(r)}\} = \{0\}$, (A4) becomes

$$\delta U_i^{(r)} = \{f_i^{(r)}\}^T\{\delta u_i^{(r)}\}/G^{(r)}, \tag{A5}$$

and from (A3) we get

$$\{\delta f_i^{(r)}\} = [\delta K_i]\{u_i^{(r)}\}+[K_i]\{\delta u_i^{(r)}\} = \{0\}.$$

Multiplying the last equation by $\{u_i^{(r)}\}^T$, we obtain

$$\{f_i^{(r)}\}^T\{\delta u_i^{(r)}\} = -\{u_i^{(r)}\}^T[\delta K_i]\{u_i^{(r)}\},$$

which, upon substitution in (A5) together with $[\delta K_i] = [K_{i,i}]\delta A_i = m[K_i]\delta A_i/A_i$, yields

$$\delta U_i^{(r)} = -mU_i^{(r)}\delta A_i/A_i. \tag{A6}$$

Expression (A6) can now be substituted in (A1), and the resulting equation solved for δA_i. The result is

$$\frac{\delta A_i}{A_i} = \frac{(m/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)}-A_i}{(m^2/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)}+A_i}. \tag{A7}$$

Since we assumed $\delta A_i/A_i$ to be small, the numerator of (A7) will also be small. Therefore, it is permissible to use the approximation

$$(m/\rho_i)\sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)} = A_i$$

in the denominator, resulting in

$$A_i' = A_i + \delta A_i = \frac{m}{m+1} A_i + \frac{(m/\rho_i) \sum_{r=1}^{\bar{R}} \lambda_r U_i^{(r)}}{m+1}. \tag{A7}$$

Equation (A7) is equivalent to the redesign formulae for active members, (17)–(19), presented previously.

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Абстракт—Работа дает итеративный метод конечного эламента для расчета констр ⁄кций на минимум веса, принимая во внимание отсутствие потери устойчивости. На основе примера оптимальности, выводится уравнение для перерасчета, в противоположности к методике численного поиска. Затем, можно трактовать задачи, которые отличаются наличием двух основных видов продольного изгиба, для оптимального расчета. Применение метода иллюстрируется задачами расчета балки и ортогональной рамы.